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The Baxter Revolution[1]

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I review the revolutionary impact Rodney Baxter has had on statistical mechanics beginning with his solution of the 8 vertex model in 1971 and the invention of corner transfer matrices in 1976 to the creation of the RSOS models in 1984 and his continuing current work on the chiral Potts model.

KEY WORDS: Baxter; corner transfer matrices; RSOS models; Potts model.

1. INTRODUCTION

At the beginning of the 20th century statistical mechanics was conceived of as a microscopic way to understand the laws of thermodynamics and the kinetic theory of gases. In practice its scope was limited to the classical ideal gas, the perfect quantum gases and finally to a diagrammatic technique devised in the 30's for computing the low density properties of gases. At that time there was even debate as to if the theory were in principle powerful enough to include phase transitions and dense liquids.

All of this changed in 1944 when Onsager[(1)] demonstrated that exact solutions of strongly interacting problems were possible by computing the free energy of the Ising model. But, while of the greatest importance in principle, this discovery did not radically alter the field of statistical mechanics in practice and relatively little progress was made in the following 25 years. However, starting with the beginning of the 70's Rodney Baxter took up the cause of exactly solvable models in statistical mechanics and from that time on the field has been so totally transformed that it may

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truly be said that a revolution has occurred. In this paper I will examine how this revolution came about.

2. THE EIGHT VERTEX MODEL

Onsager's work of 1944 was monumental but cannot be said to be revolutionary because its consequences were so extremely limited. Kaufman and Onsager[2, 3] reduced the computations to a free fermi problem in 1949 and after Yang[4] computed the spontaneous magnetization in 1952 there were no further developments. Indeed the reduction of the solution of the Ising model to a free fermi problem had the effect of suggesting that Onsager's techniques were so specialized that there might in fact not be any other statistical mechanical models which could be exactly solved.

It was therefore very important when in 1967 Lieb[5] introduced and solved (cases of) the six vertex model. This showed that other exactly solvable statistical mechanical problems did indeed exist. Lieb found that this statistical model had the very curious property that the eigenvectors of its transfer matrix were exactly the same as the eigenvectors of the quantum spin 1/2 anisotropic Heisenberg chain

$$H=-\tfrac{1}{2}\sum_{j=1}^{L}\left(\sigma_j^x\sigma_{j+1}^x+\sigma_j^y\sigma_{j+1}^y+\Delta\sigma_j^z\sigma_{j+1}^z\right) \tag{1}$$

which had been previously solved[6–9] by methods that went back to the work of Bethe[10] in 1931. This result is particularly striking because the six vertex model depends on one more parameter that does the XXZ spin chain. That extra parameter (which I will refer to as v) appears in the eigenvalues of the transfer matrix but not in the eigenvectors. The reasons for this curious relation between the quantum spin chain in one dimension and the problem in classical statistical mechanics in two dimensions were totally obscure.

At that time the author was a post doctoral fellow and he and his thesis advisor in a completely obscure paper[11] explained the relation between the quantum and classical system by demonstrating that the transfer matrix for the six vertex model $T(v)$ commutes for all v with the Hamiltonian (1) of the XXZ model.

$$[T(v), H]=0 \tag{2}$$

This commutation relation guarantees that the eigenvectors of $T(v)$ are independent of v and that they are equal to the eigenvectors of H without having to explicitly compute the eigenvectors themselves.

The next year Sutherland[12] found an identical commutation relation between the quantum Hamiltonian of the XYZ model

$$H_{XYZ}=\sum_{j=1}^{L}\left(J^x\sigma_j^x\sigma_{j+1}^x+J^y\sigma_j^y\sigma_{j+1}^y+J^z\sigma_j^z\sigma_{j+1}^z\right) \tag{3}$$

and the transfer matrix of the eight-vertex model. But since neither the eight vertex model nor the XYZ model had been solved this commutation relation merely related two equally intractable problems.

All mysteries were resolved when in 1971 Baxter solved both the eight vertex model and the XYZ model[13–15] at the same time and moreover solved them by inventing methods of such power and generality that the the course of research in statistical mechanics was permanently altered. This is the beginning of the Baxter revolution.

The first revolutionary advance made by Baxter[13–15] was the generalization of

$$[T(v), H]=0 \qquad \text{to} \quad [T(v), T(v')]=0 \tag{4}$$

and that as $v\sim 0$

$$T(v)\sim T(0)(1+cv+vH_{XYZ}) \tag{5}$$

This generalization is of great importance because it relates a model to itself and can be taken as a general criteria which selects out particular models of interest. Moreover, Baxter demonstrated the existence of this global commutation relation by means of a local relation between Boltzmann weights. Baxter called this local relation a star triangle equation because the first such relation had already been found by Onsager[1,16] in the Ising model and Onsager had referred to the relation as a star triangle equation. A related local equation had been known since the work of McGuire[17] and Yang[18] on the quantum delta function gases but its deep connection with the work of Onsager had not been understood. The search for solutions of the star triangle equation has been of major interest ever since and has led to the creation of the entirely new field of mathematics called "Quantum Groups."[19,20] The Baxter revolution of 1971 is directly responsible for this new field of mathematics.

The second revolutionary step in Baxter's paper[13] is that in addition to the commutation relation (4) he was able to obtain a functional equation for the eigenvalues of the transfer matrix and from this he could obtain equations which characterized the eigenvalues. In the limit where the eight vertex model becomes the six vertex model these equations reduced to the

Bethe's equations previously found by Lieb.[5] But Lieb found his equations by finding expressions for all of the eigenvectors of the problem whereas Baxter never considered eigenvectors at all. It is truly a revolutionary change in point of view to divorce the solution of the eigenvalue and eigenvector problems and to solve the former without knowing anything of the latter. This technique has proven to be of utmost generality and, indeed, for almost every solution which has been found to the star triangle equation a corresponding functional equation for eigenvalues has been found. On the other hand, the study of the eigenvectors, which was the heart of the solution of the six vertex and XXZ models has almost been abandoned.

The final technique introduced by Baxter is the thorough going use of elliptic functions. Elliptic functions, of course, have been used in physics since the days of the heavy symmetric top and are conspicuously used in Onsager's solution of the Ising model. But even though elliptic functions appear in Onsager's final expression for the free energy of the Ising model they play no role in either Onsager's original algebraic solution or in Kaufman's free fermi solution. On the other hand there are steps in Baxter's solution where the elliptic functions are essential. It is quite fair to say that just as Onsager invented the loop group of sl_2 in his solution of the Ising model that Baxter in his 1971 paper first introduced the use of the principles of modular invariance into physics.

3. THE CORNER TRANSFER MATRIX

It took Onsager 5 years from the computation of the Ising model free energy before he made public his conjecture for the order parameter.[21] Baxter was much more prompt in the case of the eight vertex model and produced in 1973 a conjecture for the order parameter[22] a mere two years after the free energy was computed. For the Ising model it took another three years to go from the conjecture to a proof.[4] For the eight vertex model it also took Baxter three years to obtain a proof of the conjecture.

The details of Baxter's proof are contained in two separate papers[23, 24] and form the subject of Chapter 13 of his 1982 book *Exactly Solved Models in Statistical Mechanics*.[25] It is even more revolutionary than the 1971 free energy computation. Baxter not only abandons the use of the eigenvectors of the row to row transfer matrix (which had been retained in his 1973 computation of the free energy of the six vertex model order parameter[26]) but he abandons the use of the row to row transfer matrix altogether. In its place he uses a completely new construct which had never been seen before and which had absolutely no precursors in the literature: the corner transfer matrix.

A transfer matrix builds up a large lattice one row at a time. In an $L \times L$ lattice of a 2 state per site model it has dimension 2^L. A corner transfer matrix builds up a lattice by adding one quadrant at a time and has dimension $2^{L^2/4}$. The spin whose average is being computed lies at the corner common to all four quadrants. Order parameters are computed from the eigenvector of the ground state of the row to row transfer matrix. For the corner transfer matrix the order parameter is expressed in terms of the eigenvalues and the eigenvectors are not needed.

Thus far the philosophy of the order parameter computation has followed the spirit of the free energy computation in that all attention has been moved from eigenvectors to eigenvalues. But in order to make this a useful tool Baxter takes one more revolutionary step. He takes the thermodynamic limit before he obtains equations for the eigenvalues. This is exactly the opposite from what was done in the free energy computation where the equations are obtained first and only in the end is the thermodynamic limit taken.

This early introduction of the thermodynamic limit has a very dramatic impact on the eigenvalues of the corner transfer matrix. To see this we note that the matrix elements of the corner transfer matrix are all doubly periodic functions of the spectral variable v. This is of course also true for the row to row transfer matrix. It is thus a natural argument to make to say that a matrix with doubly periodic elements should be have doubly periodic eigenvalues and this is in fact true for the row to row transfer matrix. But for the corner transfer matrix the taking of the thermodynamic limit has the astounding effect that the eigenvalues, instead of being elliptic functions all become simple exponentials $e^{-\alpha_r v}$. Once these very simple exponential expressions for the eigenvalues are obtained it is a straightforward matter to obtain the final form for the spontaneous magnetization of the eight vertex model, but all along the way, it is fair to say, a great deal of magic has been worked.

4. THE RSOS MODELS

The next stage in the Baxter revolution is the discovery and solution of the RSOS model by Andrews, Baxter and Forrester in 1984.[27] As in the case of the eight vertex model revolution in 1971 there were several precursor papers, this time all by Baxter himself.

It has been stressed in the preceding sections that Baxter made a revolutionary shift of point of view by discovering that the eigenvalue problems could be solved without solving the eigenvector problems. Therefore for the six vertex and XXZ model Baxter could obtain the Bethe's equations for the eigenvalues without recourse to the Bethe's form of the eigenvector

$$\psi(x_1, x_2,..., x_n) = \sum_P A(P)\, e^{i\sum_j x_j k_{Pj}} \tag{6}$$

In the previous work on the six vertex and XXZ models the restriction was made that all the k_j were distinct. It was therefore quite a surprise when in 1973 Baxter discovered[28] that in the XXZ chain (1) when

$$\Delta = \tfrac{1}{2}(q + q^{-1}) \qquad \text{and} \qquad q^{2N} = 1 \tag{7}$$

that there are in fact eigenvectors of the XXZ chain for which the k_j of (6) are equal. For these solutions the k_j obey

$$\Delta e^{ik_j} - 1 - e^{i(k_j + k_l)} = 0 \tag{8}$$

and this case had been tacitly excluded in all previous work.

In ref. 28 Baxter generalized the root of unity condition (7) of the six vertex model to the eight vertex model and he found an entire basis of eigenvectors which in a sense makes maximal use of the violation of the previously assumed condition $k_j \neq k_l$. Baxter is thus able to re-express these root of unity eight vertex models in terms of what he calls in his 1973 paper an "Ising-like model with a four spin interaction."

Baxter's next encounter with root of unity models was in 1981 when he solved the hard hexagon model.[29] In this most remarkable paper Baxter uses his corner transfer matrices to compute the order parameter of the problem and in the course of the computation discovers the identities of Rogers[30] and Ramanujan[31] which were first found in 1894

$$\sum_{n=0} \frac{q^{n(n+a)}}{(q)_n} = \frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} (q^{n(10n+1+2a)} - q^{(5n+2-a)(2n+1)}) \tag{9}$$

where $(q)_n = \prod_{j=1}^{n} (1 - q^j)$ and $a = 0, 1$. Baxter was clearly impressed that these classic identities appeared naturally in a statistical mechanics problem because he put the term "Rogers–Ramanujan" in the title of the paper. Because the right hand side of (9) is obviously written as the difference of two theta functions we once again see that modular functions appear naturally in statistical mechanics. But neither the 1973 nor the 1981 papers can be called genuinely revolutionary because neither of them was seen to have general applicability.

The revolution that allowed the general applicability of Baxter's techniques is carried out in the paper of 1984 with Andrews and Forrester[27] and the companion paper by Forrester and Baxter[32] in which it was shown that the hard hexagon model of ref. 29 is obtained from a special case of the "Ising-like models" found in the root of unity eight vertex models in 1973.[28] These Ising-like models are now called solid-on-solid models and the restriction needed to obtain the hard hexagon model is in the general case called the restricted solid-on-solid model. Starting from this formulation of the RSOS models the order parameters are computed by a direct application of the corner transfer matrix method and at the step where in the hard hexagon model the identity (9) was obtained the authors of refs. 27, 32 instead solve a path counting problem and find the general result

$$\frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} \left(q^{n(npp'+rp'-sp)} - q^{(np'+s)(jp+r)}\right) \tag{10}$$

where the relatively prime integers p and p' effectively parameterize the root of unity condition (7). This sum in this result is obviously the difference of two Jacobi theta functions and thus we see that all the RSOS models lead to theta functions. But most remarkably the exact same expression (10) was discovered at the same time to arise in the expression of the characters[33, 34] of the minimal models $M(p, p')$ conformal field theory[35] and these models were soon thereafter obtained as cosets[36] of the affine Lie algebra $A_1^{(1)}$.

It thus became clear that the statistical mechanics of RSOS models, conformal field theory, and affine Lie algebras are all part of the same subject and from this point forth the results of statistical mechanics appear in such apparently unrelated fields as string theory,number theory and knot theory. Baxter's corner transfer matrix was seen to be intimately related to constructions in the theory of affine Lie algebras involving null vectors and the corner transfer matrix computations of Baxter's statistical models were rapidly generalized from the affine Lie algebra $A_1^{(1)}$ to all affine Lie algebras. Solvable statistical mechanical models were now seen everywhere in physics and Baxter's methods were subject to vast generalization.

5. THE CHIRAL POTTS MODEL

For a few years it was thought that the revolution was complete and that corner transfer matrix methods and group theory could solve all problems which started out from commuting transfer matrices. This was changed however when the chiral Potts model was discovered in 1987.[37] This model does indeed satisfy the condition of commuting transfer matrices (4) and the Boltzmann weights do obey a star triangle equation but unlike all previously seen models the Boltzmann weights are not parameterized either by trigonometric or elliptic functions but rather are functions on some higher genus spectral curve. There is a modulus like variable k in the model and when $N=3$ the genus of the curve is 10 if $k \neq 0, 1$ and if $k=1$ the curve is the very symmetric elliptic curve $x^3+y^3=z^3$. If $N=4$ and $k=1$ the curve is the fourth order Fermat curve $x^4+y^4=z^4$ which has genus three.[38]

As would be expected Baxter rapidly became interested in this problem and soon Baxter, Perk and Au–Yang[39] found that for arbitrary N and k the spectral curve has the very simple form

$$a^N + kb^N = k' d^N \qquad \text{and} \qquad ka^N + b^N = k' c^N \tag{11}$$

with $k^2 + k'^2 = 1$. When $N=2$ this curve reduces to an elliptic curve and the chiral Potts model reduces to the Ising model. However, in general for $k \neq 0, 1$ the curve has genus $N^3 - 2N^2 + 1$ and for $k=1$ the curve reduces to the Nth order Fermat curve of genus $(N-1)(N-2)/2$.

The first thing to attempt after finding the Boltzmann weights for the chiral Potts model is to repeat what had been done so many times before and to obtain a functional equation for the eigenvalues. That was soon done[40–43] but the next step in Baxter's program was not so easy because the methods of solution of this functional equation which relied on the properties of genus 1 elliptic functions did not work. Solutions for the free energy which by passed the elliptic functions were soon found[40–44] but the fact that new methods were needed indicated that the revolution was not yet complete.

The greatest puzzle was set up in 1989 when after generalizing earlier work on the $N=3$ state model[45] it was conjectured[46] on the basis of extensive series expansions that the order parameters of the N state chiral Potts model are given by

$$M_n = (1-k^2)^{n(N-n)/2N^2} \qquad \text{for} \quad 1 \leqslant n \leqslant N-1 \tag{12}$$

This remarkably simple expression reduces to the result of Onsager[21] and Yang[4] for the Ising model when $N=2$ and is a great deal simpler than the order parameters for the RSOS models.[27, 32] The first expectation was that Baxter's corner transfer matrix methods could be applied to prove the conjecture true and the first attempt to do this was made by Baxter in ref. 47. In this paper Baxter gives a new and very transparent derivation of the corner transfer matrix methods and he reduces the computation of the order parameter to a problem of the evaluation of a path ordered exponential of non-commuting operators over a Riemann surface. Such a formulation sounds as if methods of non Abelian field theory could now be applied to solve the problem. Unfortunately to quote Baxter in a subsequent paper[48] "Surprisingly the method completely fails for the chiral Potts model."

The reason for the failure of the method is that the introduction of the higher genus curve into the problem has destroyed a property used by Baxter and all subsequent authors in the application of corner transfer matrix methods. This property is the so called difference property which is the property, shared by the plane and torus but by no curve of higher genus, of having an infinite automorphism group (the translations). It is this property which was used to reduce the eigenvalues to exponentials in the spectral variable and it is not present in the chiral Potts model.

6. FUTURE PROSPECTS

The discovery of the chiral Potts model has made it now clear that the Baxter revolution has met up with problems in algebraic geometry which have proven intractable for almost 150 years. Baxter has investigated these problems now for almost a decade[47–51] and it is clear that the solution of these physics problems will make a major advance in mathematics. But even with this evaluation of current problems the impact of Baxter's revolution is clearly seen. Mathematics is no longer treated as a closed finished subject by physicists. More than anyone else Baxter has taught us that physics guides mathematics and not the other way around. This is of course the way things were in the 17th century when Newton and Leibnitz invented calculus to study mechanics. Perhaps in the intervening centuries in the name of being experimental scientists we physicists drifted away from away from doing creative mathematics. The work of Rodney Baxter serves now and will serve in the future as a beacon of inspiration to all those who believe that there is a unity in physics and mathematics which provides insight that can be obtained in no other way.

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